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On Axially Symmetric Solutions of Einstein's Field Equations in Matter and an Example of Static Gravitational Shielding

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Abstract

A model system, consisting of a thin spherical shell with radius R and mass M and a point mass m at a distance $s > R$ from the center of the sphere, held fixed by an appropriate strut, is solved to order mM. The stresses in the shell are not of the canonical Weyl type, and it is argued that the same is true for more realistic situations, e.g., rotating matter. Owing to the nonlinearity of Einstein's field equations, the field of the point mass is shielded from the interior of the shell by a factor η lying between $1 - 3M/R$ and $1 - 2M/R$, and the field outside the shell explicitly depends on R.

§(1): *Introduction*

In the last decade, very powerful methods have been worked out which allow one to construct a great many axially symmetric and stationary solutions of Einstein's vacuum field equations. (See, e.g., [1-3] for an overview.) However, much less is known about realistic material sources for these vacuum solutions, and there does not seem to exist a single exact solution of Einstein's equations combining in a continuous way the interior and exterior of a realistic rotating star. Since the pioneering work of Weyl [4-6], it is known that Einstein's field equations for axially symmetric and stationary or static configurations simplify very

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much and acquire some similarity to the vacuum equations, if the energy-momentum tensor $T^{\mu}{}_{\nu}$ of the matter fulfills the "canonical" condition $T^{\rho}{}_{\rho} + T^{z}{}_{z} = 0$, where ρ and z are radial and axial coordinates. This condition is trivially valid for dust solutions and also for some types of electromagnetic fields, but of course it will not be satisfied by more general and more realistic models of matter.

As a simple and intuitive example, we consider the two-body system, consisting of a thin spherical shell and a point mass outside of this shell. An appropriate strut, connecting both bodies, maintains a static equilibrium. Configurations with thin mass shells have the privilege that, owing to the contracted Bianchi identities, the spatial stresses in these shells are uniquely determined by the energy density distribution. For our case, and, to be sure, for many other configurations, the stresses in the shell and in the strut definitely have $T^{\rho}{}_{\rho} + T^{z}{}_{z} \neq 0$.

Besides the property of being a noncanonical Weyl solution, our model system deserves interest under the aspect of gravitational shielding. There are a number of phenomena in general relativity which can be interpreted as gravitational shielding or compensation, some of which have analogs in electromagnetism, others do not: For a given gravitational field it is always possible to compensate the curvature in a small region (to "flatten" space-time) by placing three or more appropriate point masses in the vicinity of this region, a method which may have significance for precision experiments in space laboratories [7]. Quasistatic shielding occurs for instance in the tides and in other situations, where gravitational fields induce displacements within mass configurations, and in this way lead to an effective reduction of the gravitational field [8]. For systems of rapidly rotating objects, the Newtonian gravitational attraction can be reduced by a repulsive spin-spin interaction. As has been shown recently [9], there are vacuum solutions—but presumably no realistic matter solutions—where a complete balance between these two types of gravitational interactions can be reached. All of these examples of gravitational shielding or compensation are analogous to corresponding electromagnetic phenomena, differing from the electromagnetic examples only by sign and by their quadrupole instead of dipole character.

In general relativity, there exist, however, still other types of gravitational compensation and shielding, having no analog in electromagnetism. Besides the well-known possibility of local elimination of gravitational fields due to Einstein's principle of equivalence, these are brought about mainly by the nonlinearity of general relativity. And in this respect, the aforementioned system of a spherical shell and a mass point seems to be the most simple configuration, showing this shielding effect in a particularly clean way.

In Section 2, the metric and the field equations for axially symmetric and static configurations are stated, and the field equations solved in the (trivial) orders m and M. The stresses, induced in the mass shell by the point mass, are calculated to lowest order in Section 3. With these ingredients, the field equations are solved in order mM in Section 4. These solutions violate elementary flatness at parts of the axis, enforcing in this way a strut between the two bodies. In Section 5, the shielding of the point mass from the interior of the shell and the R dependence of the field outside the shell are computed and discussed.

§(2): *Metric and Field Equations for Axially Symmetric and Static Configurations*

For a very wide class of matter configurations, including our model system, a static and axially symmetric field admits 2-spaces, everywhere orthogonal to the timelike Killing vector ∂_t and to the spacelike azimuthal Killing vector ∂_ϕ [3], and since in this 2-space an "isothermal" form of the line element can be chosen [4], we will start with the following form of the metric:

$$ds^2 = -e^{2U}\, dt^2 + e^{-2U}[e^{2K}(d\rho^2 + dz^2) + F^2\, d\phi^2] \tag{1}$$

where U, K, and F are functions of ρ and z only. Herewith, Einstein's field equations $G^\mu{}_\nu = 8\pi T^\mu{}_\nu$ (with $G = c = 1$) attain a form which can be taken from many places in the literature, e.g., [10]:

$$\begin{aligned} -2U_{,\rho\rho} - 2U_{,zz} + K_{,\rho\rho} + K_{,zz} + F^{-1}(F_{,\rho\rho} + F_{,zz}) \\ + U_{,\rho}{}^2 + U_{,z}{}^2 - 2F^{-1}(U_{,\rho}F_{,\rho} + U_{,z}F_{,z}) = 8\pi e^{2(K-U)}T^t{}_t \end{aligned} \tag{2}$$

$$F^{-1}F_{,zz} - U_{,\rho}{}^2 + U_{,z}{}^2 + F^{-1}(K_{,\rho}F_{,\rho} - K_{,z}F_{,z}) = 8\pi e^{2(K-U)}T^\rho{}_\rho \tag{3}$$

$$F^{-1}F_{,\rho\rho} + U_{,\rho}{}^2 - U_{,z}{}^2 - F^{-1}(K_{,\rho}F_{,\rho} - K_{,z}F_{,z}) = 8\pi e^{2(K-U)}T^z{}_z \tag{4}$$

$$-F^{-1}F_{,\rho z} - 2U_{,\rho}U_{,z} + F^{-1}(K_{,\rho}F_{,z} + K_{,z}F_{,\rho}) = 8\pi e^{2(K-U)}T^\rho{}_z \tag{5}$$

$$K_{,\rho\rho} + K_{,zz} + U_{,\rho}{}^2 + U_{,z}{}^2 = 8\pi e^{2(K-U)}T^\phi{}_\phi \tag{6}$$

Combining these five equations, we can form the simpler equations

$$U_{,\rho\rho} + U_{,zz} + F^{-1}(U_{,\rho}F_{,\rho} + U_{,z}F_{,z}) = 4\pi e^{2(K-U)}(T^\rho{}_\rho + T^z{}_z + T^\phi{}_\phi - T^t{}_t) \tag{7}$$

$$F^{-1}(F_{,\rho\rho} + F_{,zz}) = 8\pi e^{2(K-U)}(T^\rho{}_\rho + T^z{}_z) \tag{8}$$

In Section 4 we will use these two equations to determine the functions U and F. The function K will then be found with the help of equation (4). The remaining field equations are satisfied automatically if the energy-momentum tensor $T^\mu{}_\nu$ fulfills the local conservation laws $T^\mu{}_{\nu;\mu} = 0$, which are considered in Section 3.

Since it does not seem possible to find exact solutions of these field equations for our model system of a spherical mass shell M and a point mass m, and since approximate solutions are sufficient for a physical discussion of the nonlinear shielding effect, we expand the energy-momentum tensor T and the metric functions U, K, and F in powers of m and M (resp. in powers of mG/dc^2 and

MG/dc^2, where d is a typical spatial dimension of our system as are the radius R of the spherical shell or the distance $s > R$ of the point mass from the center of the sphere), starting with Minkowski space-time ($U = K = 0, F = \rho$) in zeroth order. From these expansions, we keep only terms of order m, M, and mM. The terms $\overset{m}{T}{}^{\mu}{}_{\nu}$ and $\overset{M}{T}{}^{\mu}{}_{\nu}$ basically define our two-body system. Having a static system without energy flux, the components $T^t{}_\rho$, $T^t{}_z$, and $T^t{}_\phi$ are zero. Stresses $T^\rho{}_\rho$, $T^\rho{}_z$, etc. come about only by gravitational interaction between the mass elements of our system and therefore appear only in orders m^2, M^2, mM, and higher orders. For these reasons, in orders m and M there remain only the energy density components

$$\overset{m}{T}{}^t{}_t = -\frac{m}{2\pi\rho}\,\delta(\rho)\delta(z-s), \qquad \overset{M}{T}{}^t{}_t = -\frac{M}{4\pi R^2}\,\delta((\rho^2+z^2)^{1/2}-R) \tag{9}$$

Strictly speaking, a Schwarzschild point mass m is represented in cylindrical coordinates ρ, z, ϕ by a line of uniform mass density, extending at the axis $\rho = 0$ from $z = s - mG/c^2$ to $z = s + mG/c^2$ [4]. Differences to our ansatz (9) appear, however, only in orders m^2 and higher, which we do not consider. For the spherical mass shell M, the exact configuration has been given by Brill and Cohen [11], which, by neglecting terms of orders M^2 and higher, reduces to (9).

Now, considering the field equations in orders m and M, together with the boundary conditions for isolated systems, that U and K have to vanish asymptotically and that F has to behave asymptotically like ρ if $\rho = 0$ is chosen as the axis of the system, we find that equation (8) has the unique solution

$$\overset{(0)}{F} = \rho, \qquad \overset{(1)}{F} = 0 \tag{10}$$

Equation (7) then reduces to $\Delta U = -4\pi T^t{}_t$, which, together with (9) and the demand of continuity across the shell has the solution

$$\overset{(1)}{U} = \overset{m}{U} + \overset{M}{U} \tag{11}$$

with

$$\overset{m}{U} = -m[\rho^2 + (z-s)^2]^{-1/2}, \quad \overset{M}{U} = \begin{cases} -\dfrac{M}{R}, & \text{for } \rho^2 + z^2 \leqslant R^2 \\ -M(\rho^2+z^2)^{-1/2}, & \text{for } \rho^2 + z^2 > R^2 \end{cases} \tag{12}$$

Equation (4) then results in

$$\overset{(1)}{K} = 0 \tag{13}$$

With these solutions, the contracted Bianchi identities and therefore the remaining field equations are identically fulfilled in orders m and M.

§(3): *The Stresses in the Spherical Mass Shell*

We now come to the nontrivial part of our two-body problem, namely, the terms of order mM in the energy-momentum tensor $T^\mu{}_\nu$ and the metric functions U, K, and F. The relations $T^\mu{}_{\nu;\mu} = T^\mu{}_{\nu,\mu} + \Gamma^\mu{}_{\mu\lambda}T^\lambda{}_\nu - \Gamma^\lambda{}_{\mu\nu}T^\mu{}_\lambda = 0$ enable us to calculate the spatial stresses $\overset{mM}{T}{}^i{}_k$ in the mass shell more or less uniquely using the energy densities $\overset{m}{T}{}^t{}_t$ and $\overset{M}{T}{}^t{}_t$ and the solutions $\overset{m}{U}$ and $\overset{M}{U}$ given in Section 2.

For $\nu = t$ and $\nu = \phi$, the relations $T^\mu{}_{\nu;\mu} = 0$ are trivially satisfied owing to the time and axial symmetry of our problem. From the metric (1), it can be deduced that the only Christoffel symbols of order unity, are $\Gamma^\rho{}_{\phi\phi} \approx -\rho$, and $\Gamma^\phi{}_{\rho\phi} \approx \rho^{-1}$, and that furthermore $\Gamma^\mu{}_{\mu t} = 0$, $\Gamma^t{}_{t\rho} = U_{,\rho}$ and $\Gamma^t{}_{tz} = U_{,z}$. Therefore the relations $T^\mu{}_{\nu;\mu} = 0$ for $\nu = \rho$ and $\nu = z$ take the forms

$$\overset{mM}{T}{}^\rho{}_{\rho,\rho} + \overset{mM}{T}{}^z{}_{\rho,z} + \frac{1}{\rho}\left(\overset{mM}{T}{}^\rho{}_\rho - \overset{mM}{T}{}^\phi{}_\phi\right) = \overset{m}{U}_{,\rho}\overset{M}{T}{}^t{}_t + \overset{M}{U}_{,\rho}\overset{m}{T}{}^t{}_t \tag{14}$$

$$\overset{mM}{T}{}^\rho{}_{z,\rho} + \overset{mM}{T}{}^z{}_{z,z} + \frac{1}{\rho}\overset{mM}{T}{}^\rho{}_z = \overset{m}{U}_{,z}\overset{M}{T}{}^t{}_t + \overset{M}{U}_{,z}\overset{m}{T}{}^t{}_t \tag{15}$$

Being interested in the stresses in the mass shell, we look only for terms being singular at $(\rho^2 + z^2)^{1/2} = R$, i.e., we set $\overset{mM}{T}{}^i{}_k = f^i{}_k\delta((\rho^2 + z^2)^{1/2} - R)$ and omit the terms $\overset{m}{T}{}^t{}_t$. (The term $\overset{M}{U}_{,\rho}\overset{m}{T}{}^t{}_t$ gives no contribution anyway, and the term $\overset{M}{U}_{,z}\overset{m}{T}{}^t{}_t$ is connected with the energy-momentum tensor of the strut.) For the detailed discussion of equations (14) and (15), it is appropriate to turn to spherical coordinates r, θ by $\rho = r\sin\theta$, $z = r\cos\theta$. Equations (14) and (15) then have terms proportional to $\delta(r - R)$ and those proportional to $(d/dr)\delta(r - R)$, which have to vanish separately, in this way producing four equations (the derivatives of $f^i{}_k$ with respect to their single variable θ being denoted by $'$):

$$\sin\theta f^\rho{}_\rho + \cos\theta f^\rho{}_z = 0 \tag{16}$$

$$\cos\theta f^\rho{}_\rho{}' + \frac{1}{\sin\theta}(f^\rho{}_\rho - f^\phi{}_\phi) - \sin\theta f^\rho{}_z{}' = -\frac{mM}{4\pi}\sin\theta(R^2 - 2Rs\cos\theta + s^2)^{-3/2} \tag{17}$$

$$\sin\theta f^\rho{}_z + \cos\theta f^z{}_z = 0 \tag{18}$$

$$\cos\theta f^\rho{}_z{}' + \frac{1}{\sin\theta}f^\rho{}_z - \sin\theta f^z{}_z{}' = \frac{mM}{4\pi}\left(\frac{s}{R} - \cos\theta\right) \cdot (R^2 - 2Rs\cos\theta + s^2)^{-3/2} \tag{19}$$

Owing to (16) and (18), in the (ρ, z)-2-space $f^i{}_k$ has the structure

$$f^i{}_k = \begin{pmatrix} \cot^2\theta & -\cot\theta \\ -\cot\theta & 1 \end{pmatrix} f^z{}_z \tag{20}$$

and therefore, in the (r, θ)-2-space, only $f^\theta{}_\theta$ is nonvanishing, i.e., in the mass shell only stresses along the surface and no radial stresses exist, as is intuitively plausible.

Furthermore, (16) and (18) lead to

$$f^\rho{}_\rho + f^z{}_z = \frac{1}{\sin^2\theta} f^z{}_z \tag{21}$$

and this cannot be identically zero, because otherwise also $f^z{}_z$ and $f^\rho{}_z$ would be identically zero in contradiction to (19). Thus we see that the stresses, induced in the mass shell by the point mass m, have necessarily $T^\rho{}_\rho + T^z{}_z \neq 0$, therefore equation (8) leads to a function $\overset{mM}{F} \neq 0$, and the metric (1) does not have the canonical Weyl form [4], which is presupposed in most examples of axially symmetric and static or stationary configurations, discussed in the literature till now [3]. It might be argued that our two-body system, forced to be static by an appropriate strut, is quite unrealistic and therefore the result $F \neq \rho$ is not representative of other systems. However, since $F = \rho$ is generally possible only with $T^\rho{}_\rho + T^z{}_z = 0$, and since this is a quite unphysical condition, being valid only for dust solutions, special types of electromagnetic fields, or appropriate mixtures of electromagnetic fields and fluids, we should like to argue that for realistic material models for, e.g., rotating stars, noncanonical metrics with $F \neq \rho$ have to be taken. We should like to come back to this point in Section 4 after the explicit calculation of the metric function $\overset{mM}{F}$.

In order to calculate the stresses $\overset{mM}{T^i{}_k}$ in the mass shell explicitly, we insert (18) into (19), and obtain

$$f^z{}_z{}' = \frac{mM}{4\pi} \sin\theta \left(\cos\theta - \frac{s}{R}\right) (R^2 - 2Rs\cos\theta + s^2)^{-3/2} \tag{22}$$

with the solution

$$f^z{}_z = \frac{mM}{4\pi s^2} \left[\left(\frac{s}{R}\cos\theta - 1\right)(R^2 - 2Rs\cos\theta + s^2)^{-1/2} + \frac{C}{R}\right] \tag{23}$$

with an integration constant C, whose physical significance will be discussed in Section 4. $f^\rho{}_\rho$ and $f^\rho{}_z$ are then given by (20). $f^\phi{}_\phi$ is calculated from (17) to

$$f^\phi{}_\phi = \frac{mM}{4\pi}\left(1 - \frac{s}{R}\cos\theta\right)(R^2 - 2Rs\cos\theta + s^2)^{-3/2} - (f^\rho{}_\rho + f^z{}_z) \tag{24}$$

The singularities of $f^\rho{}_\rho, f^\rho{}_z$, and $f^\phi{}_\phi$ at $\theta = 0$ and $\theta = \pi$ are to be expected physically, because at these points the strut impinges upon the mass shell, as outlined in Section 4.

§(4): *Solution of the Field Equations in Order mM*

With the spatial stresses $\overset{mM}{T^i{}_k}$ known, we can now set to work to solve the field equations (7), (8), and (4) in order mM. In this order, the field equations reduce, due to (10) and (13), to

$$\Delta \overset{mM}{U} = 4\pi(\overset{mM}{T^\rho{}_\rho} + \overset{mM}{T^z{}_z} + \overset{mM}{T^\phi{}_\phi} - \overset{mM}{T^t{}_t}) + 8\pi(\overset{m}{U}\overset{M}{T^t{}_t} + \overset{M}{U}\overset{m}{T^t{}_t}) \tag{25}$$

$$\overset{mM}{F}_{,\rho\rho} + \overset{mM}{F}_{,zz} = 8\pi\rho(\overset{mM}{T^\rho{}_\rho} + \overset{mM}{T^z{}_z}) \tag{26}$$

$$\overset{mM}{K}_{,\rho} = \overset{mM}{F}_{,\rho\rho} + 2\rho(\overset{m}{U}_{,\rho}\overset{M}{U}_{,\rho} - \overset{m}{U}_{,z}\overset{M}{U}_{,z}) - 8\pi\rho\,\overset{mM}{T^z{}_z} \tag{27}$$

Starting with (25), we see from (24) that the sum of the spatial stresses has the relatively simple form

$$\overset{mM}{T^\rho{}_\rho} + \overset{mM}{T^z{}_z} + \overset{mM}{T^\phi{}_\phi} = \frac{mM}{4\pi}\left(1 - \frac{s}{R}\cos\theta\right)(R^2 - 2Rs\cos\theta + s^2)^{-3/2}\delta(r - R) \tag{28}$$

which is, moreover, independent of the integration constant C.

The term $\overset{mM}{T^t{}_t}$ is however as yet unknown, and for instance not fixed by the energy-momentum conservation laws $T^\mu{}_{\nu;\mu} = 0$. Rather, $\overset{mM}{T^t{}_t}$ serves to define (in order mM) the physical system, we are considering. Since, for static systems, $T^t{}_t$ represents the invariant rest mass density, it should be constant on the shell. (A nonuniform mass distribution on the shell could simulate a shielding effect even in Newtonian theory of gravitation!) Furthermore, there should be a symmetry between the action of m on the shell M and the back-reaction of M on m, so that, in analogy to (9), we unambiguously come to the ansatz

$$\overset{mM}{T^t{}_t} = \beta\,\frac{mM}{4\pi s}\left[\frac{2}{\rho}\,\delta(\rho)\delta(z - s) + \frac{1}{R^2}\,\delta(r - R)\right] \tag{29}$$

The constant β has to be fixed by the total energy of our two-body-system, which we wish (in our approximation) to be equal to

$$E_{\text{tot}} = m + M - \frac{mM}{s} \tag{30}$$

This value can be justified for instance by the following prescription for installing our two-body system: we start with the two bodies at very large distance ($s \to \infty$), where the total energy surely is $m + M$. We tie the bodies by some sort of wires to springs which stay at infinity. By the gravitational attraction the bodies get into (slow!) motion towards each other, rolling up the wires and stretching the springs, until the process stops at some finite distance s of the bodies. The energy lost by the two body system is stored in the springs, and for $s \gg m, M$ this energy loss is, in analogy to Newtonian gravity, approximately equal to mM/s, as could in principle be tested experimentally. (As is well known from the literature and discussed at the end of this section, the wires or struts, which hold the bodies in static equilibrium finally, do not contribute to E_{tot}.)

In order to calculate the constant β from the prescription (30), we recall that E_{tot} is given by the asymptotic behavior of the metric functions for $r \to \infty$, and in this regime the contributions from $\overset{mM}{K}$ and $\overset{mM}{F}$ are negligible compared to the contributions from $\overset{mM}{U}$, as can be read off from the explicit expressions for these functions given later on in this section. Considering now equation (25) in the asymptotic region $r \to \infty$, only the volume integrals of the sources (the monopole terms) are relevant. The volume integral of (28) vanishes however, as is also intuitively clear, since the stresses in the mass shell have to compensate each other as a whole. There remains the asymptotic equation

$$\Delta \overset{mM}{U}_{as} = 8\pi\,(\overset{mM}{UT^t{}_t} + \overset{Mm}{UT^t{}_t}) - 4\pi\,\overset{mM}{T^t{}_t}$$

which, in view of (9), (12), and (29), has the asymptotic solution

$$\overset{mM}{U}_{as} = (2\beta - 4)\,\frac{mM}{sr}$$

so that the total U has the asymptotic form

$$(\overset{m}{U} + \overset{M}{U} + \overset{mM}{U})_{as} = -\frac{1}{r}\left[m + M - (2\beta - 4)\,\frac{mM}{s}\right]$$

Comparison with (30) leads to $\beta = \frac{5}{2}$. (At the end, we will see that the value of β does not have much influence on the shielding phenomenon.)

The complete equation (25) now reads

$$\begin{aligned}\Delta \overset{mM}{U} = mM\Bigg[&-\frac{1}{s\rho}\,\delta(\rho)\,\delta(z - s) - \frac{5}{2sR^2}\,\delta(r - R)\\ &+ \frac{5}{2R^2}\,(R^2 - 2Rs\cos\theta + s^2)^{-1/2}\delta(r - R)\\ &- \frac{s^2 - R^2}{2R^2}\,(R^2 - 2Rs\cos\theta + s^2)^{-3/2}\delta(r - R)\Bigg]\end{aligned} \tag{31}$$

The first term of the right-hand side leads to

$$\overset{mM}{U}{}^{(1)} = \frac{mM}{2s}\,(r^2 - 2rs\cos\theta + s^2)^{-1/2} \tag{32}$$

the second term to

$$\overset{mM}{U}{}^{(2)} = \begin{cases} \dfrac{5}{2s}\,\dfrac{mM}{R} & \text{for } r \leqslant R \\[2ex] \dfrac{5}{2s}\,\dfrac{mM}{r} & \text{for } r > R \end{cases} \tag{33}$$

The part $\overset{mM}{U}{}^{(3)}$, generated by the last two source terms of (31), could in principle be found by integration over the Green's function of the Laplacian. This leads however to elliptic integrals, which are difficult to manage and to interpret. We therefore prefer to expand $\overset{mM}{U}{}^{(3)}$ in Legendre polynomials:

$$\overset{mM}{U}{}^{(3)} = \sum_{l=0}^{\infty} f_l(r) P_l(\cos\theta)$$

with [12]

$$(R^2 - 2Rs\cos\theta + s^2)^{-1/2} = \frac{1}{s}\sum_{l=0}^{\infty}\left(\frac{R}{s}\right)^l P_l(\cos\theta)$$

$$(R^2 - 2Rs\cos\theta + s^2)^{-3/2} = \frac{1}{s(s^2 - R^2)}\sum_{l=0}^{\infty}(2l+1)\left(\frac{R}{s}\right)^l P_l(\cos\theta)$$

We obtain from (31) the following differential equations for the functions $f_l(r)$:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) f_l(r) - \frac{l(l+1)}{r^2} f_l(r) = -\frac{mM}{R^2 s}(l-2)\left(\frac{R}{s}\right)^l \delta(r-R) \tag{34}$$

In order that $\overset{mM}{U}{}^{(3)}$ falls off for $r \to \infty$ and is regular at $r = 0$, the solutions $f_l^{(a)}(r) = c_l^{(a)} r^l$ have to be taken for $r \leqslant R$, and the solutions $f_l^{(b)}(r) = c_l^{(b)} r^{-l-1}$ for $r > R$. The coefficients $c_l^{(a)}$ and $c_l^{(b)}$ are determined by the requirement that $f_l(r)$ be continuous across the shell and $f_{l,r}(r)$ has the correct discontinuity at $r = R$ according to the right-hand side of (34).

$$\overset{mM}{U}{}^{(3)}(r,\theta) = \begin{cases} \dfrac{mM}{2R}\,[\rho^2 + (z-s)^2]^{-1/2} - \dfrac{5mM}{2Rs}\displaystyle\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(\frac{r}{s}\right)^l P_l(\cos\theta), \\ \qquad \text{for } r \leqslant R \\ \dfrac{mM}{2s}\left[\rho^2 + \left(z - \dfrac{R^2}{s}\right)^2\right]^{-1/2} - \dfrac{5mM}{2Rs}\displaystyle\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(\frac{R^2}{rs}\right)^l P_l(\cos\theta), \\ \qquad \text{for } r > R \end{cases} \tag{35}$$

The first terms in these expressions look as if they were produced by point sources lying on the axis $\rho = 0$ at $z = s$ (compare with $\overset{mM}{U}{}^{(1)}$!), resp. at the "mirror" point $z = R^2/s$. The whole contribution to the function U in order mM is of course the sum

$$\overset{mM}{U} = \overset{mM}{U}{}^{(1)} + \overset{mM}{U}{}^{(2)} + \overset{mM}{U}{}^{(3)} \tag{36}$$

We now come to the solution of equation (26) for the function $\overset{mM}{F}$. Because of the factor $\delta(r - R)$ of the source term, it is advantageous to go over to spherical coordinates r, θ:

$$\overset{mM}{F}_{,rr} + \frac{1}{r}\overset{mM}{F}_{,r} + \frac{1}{r^2}\overset{mM}{F}_{,\theta\theta} = 8\pi\rho(\overset{mM}{T^\rho}_\rho + \overset{mM}{T^z}_z) \tag{37}$$

Expansion of $\overset{mM}{F}$ in the eigenfunctions $\sin l\theta$ and $\cos l\theta$ of the operator $\partial^2/\partial\theta^2$ leads to a separation of the variables r and θ. We have to demand that $\overset{mM}{F}$ is a unique function of ρ and z, and that in the expansion $\overset{(0)}{F} + \overset{(1)}{F} + \overset{mM}{F} = \rho(1 + \rho^{-1}\overset{mM}{F})$ the term $\rho^{-1}\overset{mM}{F}$ is nonsingular at the axis $\rho = 0$. Therefore l has to be an integer, and only the eigenfunctions $\sin l\theta$ are allowed:

$$\overset{mM}{F} = \sum_{l=0}^{\infty} g_l(r)\sin(l+1)\theta \tag{38}$$

A corresponding expansion of the right-hand side of (37)

$$8\pi\rho(\overset{mM}{T^\rho}_\rho + \overset{mM}{T^z}_z) = \sum_{l=0}^{\infty} t_l \sin(l+1)\theta\delta(r - R) \tag{39}$$

has, with (21) and (23) the coefficients

$$t_l = \frac{4mM}{\pi s^2}\int_0^\pi d\theta\,\frac{\sin(l+1)\theta}{\sin\theta}\,[(s\cos\theta - R)(R^2 - 2Rs\cos\theta + s^2)^{-1/2} + C] \tag{40}$$

which cannot be integrated in an elementary way. Equation (37) reduces now to the following differential equations for the functions $g_l(r)$:

$$g_{l,rr} + \frac{1}{r}g_{l,r} - \frac{(l+1)^2}{r^2}g_l = t_l\delta(r - R) \tag{41}$$

The solutions, which lead to a function $\overset{mM}{F}$, falling off asymptotically and being regular at $r = 0$, are $g_l^{(a)}(r) = e_l^{(a)} r^{l+1}$ for $r \leqslant R$, and $g_l^{(b)}(r) = e_l^{(b)} r^{-l-1}$ for $r > R$. The coefficients $e_l^{(a)}$ and $e_l^{(b)}$ are again determined by continuity of g_l across the shell and correct discontinuity of $g_{l,r}$ according to (41). In this way we arrive at the following expression for the function $\overset{mM}{F}$:

$$\overset{mM}{F}(r,\theta) = \begin{cases} -\dfrac{r}{2} \displaystyle\sum_{l=0}^{\infty} \dfrac{t_l}{l+1} \left(\dfrac{r}{R}\right)^{l} \sin(l+1)\theta, & \text{for } r \leqslant R \\ -\dfrac{r}{2} \displaystyle\sum_{l=0}^{\infty} \dfrac{t_l}{l+1} \left(\dfrac{r}{R}\right)^{-l-2} \sin(l+1)\theta, & \text{for } r > R \end{cases} \tag{42}$$

The detailed form of $\overset{mM}{F}(r,\theta)$, i.e., the integration of (40) or the summation of the series in (42), is not important, since as will be shown in Section 5, $\overset{mM}{F}$ does not occur in the shielding factor. We only want to stress that $\overset{mM}{F}$ falls off like r^{-1} for $r \to \infty$, and therefore the "correction term" $\rho^{-1}\overset{mM}{F}$ falls off like r^{-2}.

If we insist on a canonical Weyl metric with $F = \rho$ and therefore $\overset{mM}{F} = 0$ in the outside region $r > R$, as is of course possible in this vacuum region and is the practice in the literature, we have to choose $g_l(r) = 0$ for $r > R$ and all l. If we now, however, wish to extend this solution in a continuous manner to the interior of the shell—as is necessary, if we look for a complete description of our physical system and not only of parts of it—we have contributions from both classes of functions $g_l^{(a)}(r) \sim r^{l+1}$ and $g_l^{(b)}(r) \sim r^{-l-1}$ in the interior, and therefore $\overset{mM}{F}$ is singular at the center of the shell $r = 0$. (Likewise, we encounter a singularity of $\overset{mM}{F}$ at $r \to \infty$ if we insist on having $F = \rho$ inside the mass shell.)

If one would even insist on having a canonical Weyl metric both outside and inside the mass shell in spite of the fact that $T^\rho{}_\rho + T^z{}_z \neq 0$ in the shell, one would be forced to give up the continuity of this coordinate system (with respect to our noncanonical Weyl coordinates ρ, z, resp. the spherical coordinates r, θ): By setting $F(\rho, z) = \bar{\rho}$ in the whole space-time, the conjugate variable $\bar{z}$, which forms together with $\bar{\rho}$ a Weyl-type coordinate system, would be given by $\partial\bar{z}/\partial\theta = -r\partial\bar{\rho}/\partial r = -r\partial F/\partial r$, and therefore

$$\bar{z} = \text{const} + r\cos\theta + \begin{cases} -\dfrac{r}{2} \displaystyle\sum_{l=0}^{\infty} \dfrac{t_l}{l+1} \left(\dfrac{r}{R}\right)^{l} \cos(l+1)\theta, & \text{for } r \leqslant R \\ +\dfrac{r}{2} \displaystyle\sum_{l=0}^{\infty} \dfrac{t_l}{l+1} \left(\dfrac{r}{R}\right)^{-l-2} \cos(l+1)\theta, & \text{for } r > R \end{cases}$$

which obviously is discontinuous at $r = R$. Therefore in the coordinate system

$(\bar{\rho}, \bar{z})$ the presence of the mass shell, at least as far as the trace part $T^{\rho}{}_{\rho} + T^{z}{}_{z}$ is concerned, would be "hidden" in a discontinuity of the coordinate system.

Since equation (8), which is the basis for having $F \neq \rho$ for our two-body system, is valid not only for static axially symmetric systems but equally for systems in stationary rotation [10], also the above arguments for having $F \neq \rho$ even in the vacuum regions take over to rotating shells. This is exemplified by the slowly rotating shell of Brill and Cohen [11], which, to first order in the rotation parameter, produces the Kerr metric in the outside region [13]. If one wants, however, to describe this system in Weyl-type coordinates in a continuous and singularity-free manner, one cannot take the usual canonical Weyl form of the Kerr metric [14] but has to work with some noncanonical form with $F \neq \rho$ in the outside region.

The arguments for having $F \neq \rho$ even in the vacuum region, however, break down if we consider instead of shell structures more realistic, extended bodies, because then appropriate boundary conditions, for instance, $\lim (T^{\rho}{}_{\rho} + T^{z}{}_{z}) = 0$ at the surface of the body, can at least in principle manage a continuous transition from canonical Weyl coordinates ($F = \rho$) in the outside region to noncanonical and singularity-free Weyl coordinates inside the body. This can for instance be seen by studying the exterior and interior Schwarzschild metric in Weyl-type coordinates, or by recent work of Marek and coworkers [15, 16] on sources for the Kerr metric.

There remains the solution of equation (27) for the metric function $\overset{mM}{K}$. Studying the singular terms of the type $\delta(r - R)$ in equation (26), we see that on the one hand

$$\overset{mM}{F}_{,\rho\rho} + \overset{mM}{F}_{,zz} \simeq \sin^2\theta\, \overset{mM}{F}_{,rr} + \cos^2\theta\, \overset{mM}{F}_{,rr} = \overset{mM}{F}_{,rr} \simeq \frac{1}{\sin^2\theta}\, \overset{mM}{F}_{,\rho\rho}$$

on the other hand, because of (21)

$$8\pi\rho(\overset{mM}{T^{\rho}}{}_{\rho} + \overset{mM}{T^{z}}{}_{z}) = \frac{1}{\sin^2\theta}\, 8\pi\rho\, \overset{mM}{T^{z}}{}_{z}$$

so that in equation (27) the terms of type $\delta(r - R)$, coming from $\overset{mM}{F}_{,\rho\rho}$ and $8\pi\rho\, \overset{mM}{T^{z}}{}_{z}$ cancel, and $\overset{mM}{K}$ is continuous at $r = R$. Now, we should like to integrate equation (27) for fixed z with respect to ρ from $\rho = \infty$ inwards. For $r > R$, the term $\overset{mM}{T^{z}}{}_{z}$ does not contribute, and with the help of the solutions (12) we get

$$\overset{mM}{K}(r > R) = \overset{mM}{F}_{,\rho} + \frac{2mM}{s^2}\left[(r - s\cos\theta)(r^2 - 2rs\cos\theta + s^2)^{-1/2} - 1\right]$$

where the integration constant takes care of the correct asymptotic behavior.

For $r < R$, the integral of the functions $\overset{m}{U}$ and $\overset{M}{U}$ has to be taken only until $\rho = (R^2 - z^2)^{1/2}$, because $\overset{M}{U}$ is constant for $r < R$, thus giving the contribution

$$\frac{2mM}{s^2}\left[(R - s\cos\theta)(R^2 - 2Rs\cos\theta + s^2)^{-1/2} - 1\right]$$

The term $\overset{mM}{T^z{}_z}$ now contributes

$$\frac{2mM}{s^2}\left[(s\cos\theta - R)(R^2 - 2Rs\cos\theta + s^2)^{-1/2} + C\right]$$

so that as a whole we have

$$\overset{mM}{K}(r < R) = \overset{mM}{F}_{,\rho} + \frac{2mM}{s^2}(C - 1)$$

Inserting the result (42) for $\overset{mM}{F}$, we get

$$\overset{mM}{K}(r,\theta) = \begin{cases} -\dfrac{1}{2}\displaystyle\sum_{l=0}^{\infty} t_l\left(\frac{r}{R}\right)^{l}\cos l\theta + \dfrac{2mM}{s^2}(C - 1), & \text{for } r \leqslant R \\[2ex] -\dfrac{1}{2}\displaystyle\sum_{l=0}^{\infty} t_l\left(\frac{r}{R}\right)^{-l-2}\cos(l+2)\theta + \dfrac{2mM}{s^2} & \\ \quad \cdot\left[(r - s\cos\theta)(r^2 - 2rs\cos\theta + s^2)^{-1/2} - 1\right], & \text{for } r > R \end{cases} \tag{43}$$

For $r \to \infty$, $\overset{mM}{K}$ falls off like r^{-2}. The detailed form of $\overset{mM}{K}$, i.e., the summation of the series, is again not very important for the following.

Now, although the solutions (36), (42), and (43) fulfill reasonable asymptotic conditions and are continuous at the mass shell, they are not yet completely satisfactory from the physical point of view because, as is well known since the work of Bach and Weyl [6], in all 2-body or n-body systems which are artificially made static by using the metric (1), elementary flatness is violated at parts of the axis $\rho = 0$. To see this in our case, let us consider at $z =$ const an infinitesimal circle around the axis, with $\rho = \epsilon$. In view of the metric form (1), this circle has radius

$$\Delta\rho = \int_0^{\epsilon} d\rho\, e^{K-U} \approx \epsilon\, e^{K(\rho=\epsilon, z) - U(\rho=\epsilon, z)}$$

due to the continuity of U and K. For the circumference, we get

$$\Delta u = \int_0^{2\pi} d\phi\, Fe^{-U} = 2\pi F(\rho = \epsilon, z)\, e^{-U(\rho=\epsilon, z)}$$

Therefore the ratio $\Delta u/\Delta\rho = 2\pi\epsilon^{-1}F(\rho=\epsilon,z)\,e^{-K(\rho=\epsilon,z)} \approx 2\pi[1+\epsilon^{-1}\overset{mM}{F}(\rho=\epsilon,z)][1 - \overset{mM}{K}(\rho=\epsilon,z)] \approx 2\pi[1+\epsilon^{-1}\overset{mM}{F}(\rho=\epsilon,z) - \overset{mM}{K}(\rho=\epsilon,z)]$ differs from the Euclidean value 2π if $\epsilon^{-1}\overset{mM}{F}(\rho=\epsilon,z) - \overset{mM}{K}(\rho=\epsilon,z) \neq 0$ in the limit $\epsilon \to 0$. With the results (42) and (43), we get

$$w(z) = \lim_{\epsilon\to 0}\, [\epsilon^{-1}\overset{mM}{F}(\rho=\epsilon,z) - \overset{mM}{K}(\rho=\epsilon,z)]$$

$$= \begin{cases} \dfrac{4mM}{s^2}, & \text{for } R < z < s \\[2mm] \dfrac{2mM}{s^2}(1 - C), & \text{for } |z| \leqslant R \\[2mm] 0, & \text{otherwise} \end{cases} \tag{44}$$

Israel [17] has provided a quite general analysis of line singularities in general relativity. According to his classification, the "Weyl strut" between our two bodies is a simple line source of conical type, for which a line energy-momentum-tensor $\overset{s}{T}{}^{\mu}{}_{\nu}$ can be obtained. According to his formulas we have

$$\overset{s}{T}{}^{t}{}_{t} = \overset{s}{T}{}^{z}{}_{z} = \frac{1}{8\pi}\, w(z)\, \frac{\delta(\rho)}{\rho} \tag{45}$$

with $w(z)$ from equation (44), and all other components of $\overset{s}{T}{}^{\mu}{}_{\nu}$ being zero. The field equation (7) then tells us that the strut does not contribute to the metric function U and therefore not to the overall mass of the system, as was already stated earlier and as is true for all Weyl struts [17]. According to equation (6), there is also no contribution to the function K. There is however a contribution to the metric function F, because in the strut likewise as for the stresses in the mass shell, the canonical condition $T^{\rho}{}_{\rho} + T^{z}{}_{z} = 0$ is not valid. According to equation (26), the additional contribution $\overset{s}{F}$ fulfills the equation

$$\overset{s}{F}_{,\rho\rho} + \overset{s}{F}_{,zz} = w(z)\delta(\rho)$$

whose solution with correct asymptotic behavior is

$$\frac{1}{\rho} \overset{s}{F} = \begin{cases} -w(z), & \text{for } \rho = 0 \\ 0, & \text{for } \rho > 0 \end{cases} \tag{46}$$

which then, together with (44), just repairs elementary flatness at $\rho = 0$.

Analyzing the stresses $\overset{s}{T}{}^z{}_z$, given by (45), we see that in the region $R < z < s$ between both bodies a positive stress in the z direction exists in the strut, which leads by virtue of

$$F = \int df\, \overset{s}{T}{}^z{}_z = 2\pi \int d\rho\, \rho\, \overset{s}{T}{}^z{}_z = \frac{mM}{s^2}$$

to the Newtonian force between the bodies [6]. Whether there exists also a strut within the mass shell, and what its properties are, depends on the constant C, and so by (23) and (24), is connected with the stresses in the mass shell: If a very stiff material is used for the mass shell, the stress $\overset{s}{T}{}^z{}_z = (mM/2\pi s^2)[\delta(\rho)/\rho]$, transferred through the outside strut, is totally taken up by the mass shell at the "upper pole" $z = R (C = 1)$. For material of other constitution, part of the stress will be transferred through an inside strut to the "lower pole" point $z = -R$. However, in order to guarantee positive stress in all parts of the strut and not to overcompensate the stress in the outside strut, the restriction $|C| \leqslant 1$ seems appropriate for the constant C.

It should be remarked, that the strut with positive stress in the region $R < z < s$ can be substituted by wires with negative stresses, resp. tensions, in the outside regions $z > s$ and $z < -R$, a configuration, which might even be preferred in view of our Gedanken experiment for installing the two-body system at the beginning of this section. Mathematically, this would be brought out by integrating equation (26) with the asymptotic condition $\overset{mM}{F} \to -(4mM/s^2) r \sin\theta$ for $r \to \infty$ (with the consequence that the metric is asymptotically no more Minkowskian), and thus substituting the "strut function" $w(z)$ from equation (44) by

$$\tilde{w}(z) = \begin{cases} 0, & \text{for } R < z < s \\ -\dfrac{2mM}{s^2}(1 + C), & \text{for } |z| \leqslant R \\ -\dfrac{4mM}{s^2}, & \text{for } z > s \text{ and } z < -R \end{cases}$$

This substitution has moreover the advantage that the energy density $\overset{s}{T}{}^{tt} \approx -\overset{s}{T}{}^t{}_t = -\overset{s}{T}{}^z{}_z$, which is always negative for Weyl struts [17], becomes positive everywhere in the outside wires.

§(5): *Discussion of the Shielding Effect*

With the metric functions U, K, and F known to order mM from Section 4, we can now compare gravitational effects inside the mass shell, as they are produced by the point mass m alone with the same effects produced by the (nonlinearly) combined action of both masses m and M, and we can in this way see whether a reduction of some or all of these effects is caused by the mass shell. In order to guarantee that we study real physical effects and are not misled by coordinate effects, we should like to consider the curvature invariants. Since inside the mass shell we have, apart from the strut on the axis, a vacuum field, the Riemann tensor $R^{\mu\nu}{}_{\lambda\kappa}$ reduces to the Weyl tensor $C^{\mu\nu}{}_{\lambda\kappa}$. We find it advantageous to use instead of the otherwise popular eigenvalues of $C^{\mu\nu}{}_{\lambda\kappa}$ the polynomial invariants of the Weyl tensor, given for instance by Weinberg [18]:

$$\begin{aligned} &C_1 = C^{\mu\nu}{}_{\rho\sigma}C^{\rho\sigma}{}_{\mu\nu}, \qquad C_2 = C^{\mu\nu}{}_{\rho\sigma}C^{\rho\sigma}{}_{\lambda\kappa}C^{\lambda\kappa}{}_{\mu\nu} \\ &C_3 = g^{-1/2}\epsilon^{\mu\nu\rho\sigma}C^{\lambda\kappa}{}_{\rho\sigma}C_{\mu\nu\lambda\kappa}, \qquad C_4 = g^{-1/2}\epsilon^{\mu\nu\rho\sigma}C^{\lambda\kappa}{}_{\rho\sigma}C^{\tau\zeta}{}_{\lambda\kappa}C_{\mu\nu\tau\zeta} \end{aligned} \tag{47}$$

For our metric (1), the nonvanishing components of the Weyl tensor can be calculated to be

$$\begin{aligned} C^{t\rho}{}_{t\rho} &= C^{z\phi}{}_{z\phi} = e^{2(U-K)}(-U_{,\rho\rho} - 2U_{,\rho}{}^2 + U_{,z}{}^2 + U_{,\rho}K_{,\rho} - U_{,z}K_{,z}) \\ C^{tz}{}_{tz} &= C^{\rho\phi}{}_{\rho\phi} = e^{2(U-K)}(-U_{,zz} + U_{,\rho}{}^2 - 2U_{,z}{}^2 - U_{,\rho}K_{,\rho} + U_{,z}K_{,z}) \\ C^{t\phi}{}_{t\phi} &= C^{\rho z}{}_{\rho z} = -(C^{t\rho}{}_{t\rho} + C^{tz}{}_{tz}) \\ C^{t\rho}{}_{tz} &= -C^{\rho\phi}{}_{z\phi} = e^{2(U-K)}(-U_{,\rho z} - 3U_{,\rho}U_{,z} + U_{,\rho}K_{,z} + U_{,z}K_{,\rho}) \end{aligned} \tag{48}$$

We see that the metric function F cancels out of all components, in agreement with the general statement by Weyl [4] that in vacuum the metric can be reduced to the "canonical" form. We see further that in all nonvanishing components of $C^{\mu\nu}{}_{\lambda\kappa}$ the index t (and equally the index ϕ) occurs either in both the upper and lower pair or in no pair. Since however in the totally antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$, the index t can occur only once, the invariants C_3 and C_4 are identically zero.

If we now look for the expansion of the components $C^{\mu\nu}{}_{\lambda\kappa}$ of equations (48) in powers of m and M, it is clear that inside the mass shell no term of order M can be present. In detail, we get, by using the vacuum field equations

$$C^{t\rho}{}_{t\rho} = C^{z\phi}{}_{z\phi} \approx -\overset{m}{U}_{,\rho\rho}\left(1 + 2\overset{m}{U} + \frac{\overset{mM}{U}_{,\rho\rho}}{\overset{m}{U}_{,\rho\rho}}\right)$$

$$C^{tz}{}_{tz} = C^{\rho\phi}{}_{\rho\phi} \approx -\overset{m}{U}_{,zz}\left(1 + 2\overset{M}{U} + \frac{\overset{mM}{U}_{,zz}}{\overset{m}{U}_{,zz}}\right) \tag{49}$$

$$C^{t\rho}{}_{tz} = -C^{\rho\phi}{}_{z\phi} \approx \overset{m}{U}_{,\rho z}\left(1 + 2\overset{M}{U} + \frac{\overset{mM}{U}_{,\rho z}}{\overset{m}{U}_{,\rho z}}\right)$$

These results show, that in our approximation also the function K cancels out and with it all dependence on the integration constant C. By inserting the expressions (49) into the invariants (47), we get by tedious but straightforward calculations:

$$C_1 = 16(1 + 4\overset{M}{U})\left(\overset{m}{U}_{,\rho z}{}^2 - \overset{m}{U}_{,\rho\rho}\overset{m}{U}_{,zz} + \frac{1}{\rho^2}\overset{m}{U}_{,\rho}{}^2\right)$$

$$+ 16\left(\overset{m}{U}_{,\rho\rho}\overset{mM}{U}_{,\rho\rho} + \overset{m}{U}_{,zz}\overset{mM}{U}_{,zz} + 2\overset{m}{U}_{,\rho z}\overset{mM}{U}_{,\rho z} + \frac{1}{\rho^2}\overset{m}{U}_{,\rho}\overset{mM}{U}_{,\rho}\right) \tag{50}$$

$$C_2 = -48(1 + 6\overset{M}{U})\frac{1}{\rho}\overset{m}{U}_{,\rho}(\overset{m}{U}_{,\rho\rho}\overset{m}{U}_{,zz} - \overset{m}{U}_{,\rho z}{}^2) - 48\left(\overset{m}{U}_{,\rho\rho}{}^2\overset{mM}{U}_{,\rho\rho}\right.$$

$$\left. + \overset{m}{U}_{,zz}{}^2\overset{mM}{U}_{,zz} + \frac{1}{\rho^3}\overset{m}{U}_{,\rho}{}^2\overset{mM}{U}_{,\rho} - \frac{1}{\rho}\overset{m}{U}_{,\rho z}{}^2\overset{mM}{U}_{,\rho} - \frac{2}{\rho}\overset{m}{U}_{,\rho}\overset{m}{U}_{,\rho z}\overset{mM}{U}_{,\rho z}\right) \tag{51}$$

Inserting the expressions (12) for $\overset{m}{U}$ and $\overset{M}{U}$, again using the vacuum field equations for $\overset{mM}{U}$, and going over to the entities $D_1 = (C_1/48)^{1/2}$ and $D_2 = (C_2/96)^{1/3}$, which in contrast to C_1 and C_2, begin with order m terms, we find that in our approximation within the mass shell the relation $D_1 = D_2 =: D$ is valid, which expresses a kind of approximate symmetry beyond the time and axial symmetries, and which in the following will make its geometrical appearance in curves within the mass shell, on which the shielding factor η is constant. The invariant D has now the following form:

$$D = m[\rho^2 + (z - s)^2]^{-3/2}\left\{1 - \frac{2M}{R} - \frac{[\rho^2 + (z - s)^2]^{1/2}}{2m}\right.$$

$$\left. \cdot\,(\rho^2\overset{mM}{U}_{,\rho\rho} + (z - s)^2\overset{mM}{U}_{,zz} + 2\rho(z - s)\overset{mM}{U}_{,\rho z})\right\} \tag{52}$$

Inserting our results (32), (33), (35), and (36) for $\overset{mM}{U}$, we can (again by quite tedious calculation) evaluate the derivatives of $\overset{mM}{U}$, with the result

$$D = m[\rho^2 + (z - s)^2]^{-3/2}\eta \tag{53}$$

with the shielding factor

$$\eta = 1 - \frac{5M}{2R} - \frac{M}{2s} + \frac{15M}{2Rs}(r^2 - 2rs\cos\theta + s^2)^{1/2} \cdot \sum_{l=0}^{\infty} \frac{P_l(\cos\theta)}{(2l+1)(2l+3)(2l+5)}\left(\frac{r}{s}\right)^l \tag{54}$$

A reasonable write-up of the shielding factor is

$$\eta = 1 - \frac{2M}{R}\zeta \tag{55}$$

with

$$\zeta = \frac{5}{4} + \frac{1}{4}\frac{R}{s} - \frac{1}{4} f\left(\frac{r}{s}, \theta\right) \tag{56}$$

and

$$f\left(\frac{r}{s}, \theta\right) = \left(\frac{r^2}{s^2} - 2\frac{r}{s}\cos\theta + 1\right)^{1/2} \sum_{l=0}^{\infty} \frac{15P_l(\cos\theta)}{(2l+1)(2l+3)(2l+5)}\left(\frac{r}{s}\right)^l \tag{57}$$

Because of the denominator $(2l + 1)(2l + 3)(2l + 5)$ and the factor $(r/s)^l$, the series in f converges quite well, and to reach 1% precision, one has to sum at most until $l = 12$. In this way one finds that in the extreme case $r/s = 1$ (mass m sitting on the top of the shell M, invariant D taken just under the shell) f varies between $f = 0$ (for $\theta = 0$) and $f \approx 1.78$ (for $\theta = \pi$); for the other extreme case $r/s \to 0$ (m infinitely far from M or D taken at the origin) we have $f = 1$ independent of θ, and therefore $\zeta = 1 + (1/4)(R/s)$ and $\eta = 1 - 2M/R - M/2s$, so that in any case η lies in the region $1 - 3M/R \leqslant \eta < 1 - 2M/R$. For $s = R$ and $s = 2R$, a pictorial presentation of the shielding factor η, resp. ζ, is given by drawing lines of constant ζ in the figures 1 and 2. The lines of constant ζ are not exactly (Euclidean) circles around the point mass m, but are a little bit prolate: The point where a curve ζ = const meets the axis has a somewhat larger (Euclidean) distance from m than the points where the same curve meets the mass shell.

It is worthwhile to figure out from what physical effects the different contributions to the shielding factor, say, ζ in (56), are coming: The main contribution $\zeta_1 = 1$, which also survives in the limit $s \to \infty$, is due to the factor e^{2U} in the components of the Weyl tensor in (48), and as such it is a typically nonlinear effect of Einstein's theory of gravitation. All other contributions are generated by the correction term $\overset{mM}{U}$ in the "Newtonian potential" U, and in detail a con-

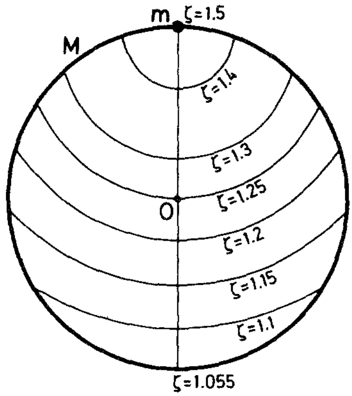

Fig. 1. The lines of constant shielding factor η, resp. ζ, in the case $s = R$.

tribution $\zeta_2 = -R/s$ comes from $\overset{Mm}{U T^t{}_t}$ in equation (25), $\zeta_3 = -(1/5)f$ from $\overset{mM}{U T^t{}_t}$, $\zeta_4 = (5/4)(R/s)$ from the term $\overset{mM}{T^t{}_t}$, $\zeta_5 = 1/4 - (1/20)f$ from the stresses $\overset{mM}{T^\rho{}_\rho} + \overset{mM}{T^z{}_z} + \overset{mM}{T^\phi{}_\phi}$. The "potential energies" $\overset{Mm}{U T^t{}_t}$, $\overset{mM}{U T^t{}_t}$ and $\overset{mM}{T^t{}_t}$ and the stresses are of course also present in Newtonian theory of gravitation, but there they do not play the role of sources for gravitational fields, and therefore no shielding is in-

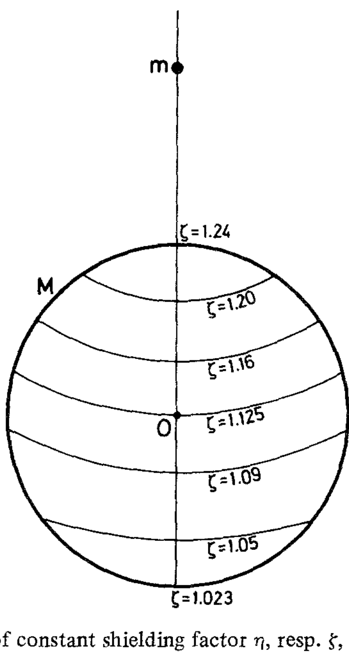

Fig. 2. The lines of constant shielding factor η, resp. ζ, in the case $s = 2R$.

duced by them. We see also that only the term ζ_4 depends on the constant β, introduced in connection with the overall energy of our system at the beginning of Section 4. For general β, we would have $\zeta_4 = \beta R/2s$, and therefore $\zeta = 5/4 - (1/4)f + (\beta/2 - 1)(R/s)$, which leads to $\eta = 1 - 2M/R$ for $s \to \infty$, independent of β.

The shielding factor η is of course relevant not only for a point mass m but for arbitrary mass distributions at distance s from the center of the shell, and for mass distributions at varying distances from the shell the factor η varies only between $1 - 2M/R$ and $1 - 3M/R$. In a similar way we can substitute for the infinitesimally thin mass shell an arbitrary spherical mass distribution with a hole at the center. Looking at such mass distributions as built up of appropriate concentric mass shells, we get the shielding factor by integrating $\eta(r = 0) = 1 - 2M/R - M/2s$ over R, e.g., for a homogeneous sphere of mass M and radius R we have

$$\bar{\eta} = 1 - \frac{3M}{R} - \frac{M}{2s} \tag{58}$$

From all these expressions for η it can be read off that total shielding ($\eta \approx 0$) seems possible only for the hypothetical case where the Schwarzschild radius $r_M = MG/2c^2$, corresponding to the mass M, is of the same order of magnitude as the spatial dimension R of this mass distribution. This result, for which our expansion in powers of MG/Rc^2 of course breaks down, might also have been expected on dimensional reasons, and it shows close analogy to the result of Brill and Cohen [11] that there is complete frame dragging inside a slowly rotating mass shell with radius $R = MG/2c^2$.

As far as more realistic examples of this nonlinear shielding phenomenon are concerned, it has to be said that–disregarding all problems with the strut, holding both bodies in equilibrium–in all cases the shielding factor η differs from unity only by an unmeasurably small amount: For a small hole in the center of a sphere of lead of radius $R = 10m$, we get $1 - \eta \approx 10^{-20}$, and even for a hypothetical hole in the center of the earth we have only $1 - \eta \approx 10^{-9}$.

We should like to add some remarks concerning the gravitational field resp. its invariants C_1 and C_2 outside the mass shell. Since there $\overset{M}{U}$ is no more constant, the expressions (49) for the approximate components of the Weyl tensor get more complicated:

$$\begin{aligned} C^{t\rho}{}_{t\rho} = C^{z\phi}{}_{z\phi} &= -(\overset{m}{U}_{,\rho\rho} + \overset{M}{U}_{,\rho\rho} + \overset{mM}{U}_{,\rho\rho} + 2\overset{M}{U}\overset{m}{U}_{,\rho\rho} + 2\overset{m}{U}\overset{M}{U}_{,\rho\rho} \\ &\quad + 4\overset{m}{U}_{,\rho}\overset{M}{U}_{,\rho} - 2\overset{m}{U}_{,z}\overset{M}{U}_{,z}) \\ C^{tz}{}_{tz} = C^{\rho\phi}{}_{\rho\phi} &= -(\overset{m}{U}_{,zz} + \overset{M}{U}_{,zz} + \overset{mM}{U}_{,zz} + 2\overset{M}{U}\overset{m}{U}_{,zz} + 2\overset{m}{U}\overset{M}{U}_{,zz} \end{aligned} \tag{59}$$

$$+ 4\overset{m}{U}_{,z}\overset{M}{U}_{,z} - 2\overset{m}{U}_{,\rho}\overset{M}{U}_{,\rho})$$

$$C^{t\rho}{}_{tz} = -C^{\rho\phi}{}_{z\phi} = -(\overset{m}{U}_{,\rho z} + \overset{M}{U}_{,\rho z} + \overset{mM}{U}_{,\rho z} + 2\overset{M}{U}\overset{m}{U}_{,\rho z} + 2\overset{m}{U}\overset{M}{U}_{,\rho z}$$

$$+ 3\overset{m}{U}_{,\rho}\overset{M}{U}_{,z} + 3\overset{M}{U}_{,\rho}\overset{m}{U}_{,z})$$

Again, the function K cancels out in our approximation. At first we want to concentrate on the lowest order terms in (59). Inserting these into the invariants (47) and making use of the vacuum field equations and of the abbreviation $V = \overset{m}{U} + \overset{M}{U}$, we reach

$$\frac{C_1}{16} \approx \frac{1}{\rho^2} V_{,\rho}{}^2 + V_{,\rho z}{}^2 - V_{,\rho\rho}V_{,zz} \tag{60}$$

$$\frac{C_2}{48} \approx \frac{1}{\rho} V_{,\rho}(V_{,\rho z}{}^2 - V_{,\rho\rho}V_{,zz}) \tag{61}$$

Inserting finally the explicit expressions (12) for $\overset{m}{U}$ and $\overset{M}{U}$ in the region $r > R$, the invariants have in lowest order the form

$$\frac{C_1}{48} \approx \{M(\rho^2 + z^2)^{-3/2} + m[\rho^2 + (z - s)^2]^{-3/2}\}^2$$
$$- 3mM\rho^2 s^2(\rho^2 + z^2)^{-5/2}[\rho^2 + (z - s)^2]^{-5/2} \tag{62}$$

$$\frac{C_2}{96} \approx \{M(\rho^2 + z^2)^{-3/2} + m[\rho^2 + (z - s)^2]^{-3/2}\}$$
$$\cdot\Big(\{M(\rho^2 + z^2)^{-3/2} + m[\rho^2 + (z - s)^2]^{-3/2}\}^2$$
$$- \frac{9}{2} mM\rho^2 s^2(\rho^2 + z^2)^{-5/2}[\rho^2 + (z - s)^2]^{-5/2}\Big) \tag{63}$$

from which it can be seen that, in contrast to the interior region $r < R$, outside the mass shell the invariants C_1 and C_2 are independent functions of ρ and z.

As a last point we want to show that the nonlinearities of Einstein's theory of gravitation induce a R dependence of the field outside the mass shell so that, in contrast to Newton's theory, the spherical body M does not act like a point mass at the origin. It suffices to exemplify this effect with the invariant C_1. [The invariant C_2 would show a similar behavior.] Going back to the expressions (59) for the Weyl tensor, we see that an R dependence is only contained in the terms $\overset{mM}{U}_{,\rho\rho}$, $\overset{mM}{U}_{,zz}$ and $\overset{mM}{U}_{,\rho z}$, and since $\overset{mM}{U}{}^{(1)}$ and $\overset{mM}{U}{}^{(2)}$ from equations (32) and (33) are R independent, we are even confined to $\overset{mM}{U}{}^{(3)}$ from equation (35). Expansion of $\overset{mM}{U}{}^{(3)}$ with respect to R results in

$$\overset{mM}{U}{}^{(3)} = -\frac{2mM}{sr} - \frac{mM}{3}\frac{zR^2}{s^2r^3} + O(R^6) \tag{64}$$

Omitting herefrom the R-independent term and neglecting terms of order R^6, we can for our purposes set

$$C^{t\rho}{}_{t\rho} = C^{z\phi}{}_{z\phi} = -\bar{U}_{,\rho\rho}, \qquad C^{tz}{}_{tz} = C^{\rho\phi}{}_{\rho\phi} = -\bar{U}_{,zz}, \qquad C^{t\rho}{}_{tz} = -C^{\rho\phi}{}_{z\phi} = -\bar{U}_{,\rho z}$$

with

$$\bar{U} = V - W = V - \frac{mM}{3}\frac{zR^2}{s^2r^3} \tag{65}$$

Calculating with these expressions the invariant C_1, we get

$$\frac{C_1}{16} \approx \left(\frac{1}{\rho^2}V_{,\rho}{}^2 + V_{,\rho z}{}^2 - V_{,\rho\rho}V_{,zz}\right) - \left(\frac{2}{\rho^2}V_{,\rho}W_{,\rho} + 2V_{,\rho z}W_{,\rho z} - V_{,\rho\rho}W_{,zz} - V_{,zz}W_{,\rho\rho}\right) \tag{66}$$

and explicitly

$$\begin{aligned}\frac{C_1}{48} \approx \frac{C_1(R=0)}{48} &+ mMR^2s^{-2}(\rho^2+z^2)^{-5/2} \\ &\cdot \left(2z\{M(\rho^2+z^2)^{-3/2} + m[\rho^2+(z-s)^2]^{-3/2}\}\right. \\ &\left.+ m\rho^2 s(2\rho^2+2z^2-5sz)(\rho^2+z^2)^{-1}[\rho^2+(z-s)^2]^{-5/2}\right)\end{aligned} \tag{67}$$

where for $C_1(R=0)/48$ the expression (62) can be taken in our approximation. Since this expression is quite tedious to handle in general, we specialize to positions at the axis $\rho = 0$ and z near $\pm R$, and to values $s \approx R$, where the maximal contribution of the R-dependent terms relative to the R-independent terms can be expected (as can be proven in detail). Assuming $m < M$, as is consistent with the fact that m has been handled as point mass, we get for $z \approx \pm R$

$$C_1(R) \approx C_1(R=0)\left(1 \pm \frac{2m}{R}\right) \tag{68}$$

that means a small amplification of the curvature due to the R-dependent terms in the region between the bodies and a small reduction in the region "behind" the mass shell.

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