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A shell around a black hole

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Abstract. We present an exact solution describing a black hole surrounded by a massive shell. Energy conditions for the material of the shell are examined and yield its minimum radius.

1. Introduction

The Penrose process allows the mining of energy from a rotating black hole; it would also be an excellent way of waste disposal; cf [1, p 908]. Of course, one would have to build structures near the black hole and the question is how close this could be done. A rope can be hung right down to the horizon of a Schwarzschild black hole [2, 3] if it is made of extremely stiff material, i.e. has equation of state $\mu = |p|$. This result, however, does not tell anything about how close one can build a shell around a black hole, which is what would be about the minimum requirement for the exploitation of the Penrose process.

In this paper we shall therefore study an exact solution of Einstein's equations which describes a static black hole surrounded by a concentric spherical massive shell. We confine ourselves to a static situation because for a Kerr black hole an exact shell source is not known, even though Babala [4] has given such a solution in terms of an expansion in the angular momentum.

2. The solution

To find the global solution which describes the situation we wish to study, we divide spacetime into two regions separated by the shell. In both regions the metric being spherically symmetric and static assumes the Schwarzschild form

$$\mathrm{d}s^2 = \begin{cases} \dfrac{\mathrm{d}\bar{r}^2}{1-2m/\bar{r}} + \bar{r}^2(\mathrm{d}\bar{\vartheta}^2 + \sin^2\bar{\vartheta}\,\mathrm{d}\bar{\varphi}^2) - (1-2m/\bar{r})\,\mathrm{d}\bar{t}^2 & \text{inside the shell} \\ \dfrac{\mathrm{d}r^2}{1-2M/r} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) - (1-2M/r)\,\mathrm{d}t^2 & \text{outside the shell} \end{cases}$$

M is the total mass of the system, m the one of the black hole. In order to use the same coordinate system inside and outside the shell and to make the metric continuous across the shell—which we assume to be described by an equation like $r = \text{const}$, respectively $\bar{r} = \text{const}$—we transform coordinates such that

$$\bar{\vartheta} = \vartheta \qquad \bar{\varphi} = \varphi \qquad \bar{t} = (\alpha_M/\alpha_m)t \qquad \bar{r}(r) \text{ such that } \bar{r}(R) = R$$

$$(\mathrm{d}\bar{r}/\mathrm{d}r)(R) = \alpha_m/\alpha_M$$

$$\alpha_M = (1 - 2M/R)^{1/2} \qquad \alpha_m = (1 - 2m/R)^{1/2}.$$

The shell is located at $r = R$. Of course $m, M < \frac{1}{2}R$.

3. The shell

To calculate the energy-momentum tensor of the shell we use a well known method [1, p 551]. Let $\boldsymbol{K}_i$ and $\boldsymbol{K}_0$ be the second fundamental forms of the two different imbeddings of the shell, $\boldsymbol{g}$ the metric on the shell and $\boldsymbol{\gamma} = \boldsymbol{K}_0 - \boldsymbol{K}_i$. The surface energy-momentum tensor is then given by

$$\boldsymbol{S} = (1/8\pi)(\boldsymbol{\gamma} - \boldsymbol{g}\,\mathrm{tr}(\gamma)).$$

It turns out that the pressure is isotropic which was to be expected anyway from symmetry considerations. Velocity, energy density and pressure are given by

$$\boldsymbol{u} = (1/\alpha_M)\partial_t \qquad \mu = \frac{1}{4\pi R}(\alpha_m - \alpha_M)$$

$$p = \frac{1}{8\pi R^2}\left(\frac{R-M}{\alpha_M} - \frac{R-m}{\alpha_m}\right).$$

Given μ we can define the mass of the shell in the static reference frame given by $\boldsymbol{u}$ by integrating the energy density over the shell at constant time. Hence

$$M_s = 4\pi R^2 \mu = R(\alpha_m - \alpha_M).$$

Solving this for the total mass yields

$$M = m + M_s(\alpha_m - M_s/2R).$$

This expression reflects the non-linearity of general relativity; the masses of the black hole and shell do not add up to the total mass. For $R \gg m$ we have $\alpha_m \simeq 1 - m/R$, and the above expression shows that the total mass is diminished by the binding energy of the shell and its gravitational self-energy.

We now impose the following energy conditions [5] on the material of the shell.

I. Weak energy condition: $0 \leqslant \mu, 0 \leqslant \mu + p$

II. Dominant energy condition: $|p| \leqslant \mu$

III. Timelike convergence condition: $0 \leqslant \mu + 2p$.

The usual condition $0 \leqslant \mu + 3p$ had to be replaced by III due to the two spatial dimensions of the shell.

The energy density has to be positive by I which requires $m \leqslant M$. It can then easily be verified that the pressure is positive for any value of R. The only condition remaining to be satisfied is thus II.

The equation $\mu - p = 0$ to be solved for R yields the minimum radius at which the shell can be built. Using M, m this gives a quadratic equation whose larger solution has to be taken; expressed in terms of M_s, m the equation is of third order. The minimum radius is given by

$$R_{\min} = \tfrac{25}{24}[M + m + (M^2 + m^2 - \tfrac{46}{25}Mm)^{1/2}].$$

It is easy to show that $R_{\min} > 2M$, the shell is thus outside the Schwarzschild radius of the system. Indeed, for $M_s \ll m$ we find approximately

$$R_{\min} \simeq \tfrac{5}{2}m + \tfrac{5}{4}M_s\alpha_m.$$

A light shell can thus be built inside the circular photon orbit but is bounded away from the horizon. For $m = 0$, i.e. no black hole inside the shell, we get as the minimum radius

$$R_{\min} = \tfrac{25}{12}M.$$

This should be compared with the minimum radius of a fluid ball which is $\tfrac{9}{4}M$ [6]. A shell can thus be more compact than a fluid ball of the same mass.

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